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LOW-ENERGY THEOREMS FOR QCD AT FINITE TEMPERATURE AND CHEMICAL POTENTIAL

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Abstract

The low energy theorems for QCD are generalized to finite temperature and chemical potential, including non-zero quark masses.

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Low-energy theorems for quantum chromodynamics (QCD) at zero temperature and density were derived long ago by Novikov *et al.* [1]. They are useful in a number of contexts, for instance in constraining effective theories or in assessing lattice gauge calculations. Recently the theorems were generalized to finite temperature ($T > 0$) for the pure glue sector of QCD [2]. The purpose of the present note is to provide a further generalization by including the quark sector and allowing for non-zero density or chemical potential, μ.

For clarity we shall consider one quark flavor since the case of several flavors is trivially obtained by introducing flavor-dependent masses, m_0, and chemical potentials, μ, and appropriately summing over the flavor index. In the imaginary time approach the partition function takes the familiar form

$$\begin{aligned} Z = &\int [d\bar{A}][dq][d\bar{q}] \\ &\times \exp\left\{ \int_0^{1/T} d\tau d^3x \left[-\frac{1}{4g_0^2} \bar{F}_a^{\mu\nu} \bar{F}^a_{\mu\nu} + \bar{q}(i\partial\!\!\!/ - \tfrac{1}{2}\bar{A}\!\!\!/_a \lambda^a + \mu\gamma_0 - m_0)q \right] \right\}. \end{aligned} \quad (1)$$

Here we have suppressed the gauge fixing and Fadeev-Popov ghost terms, as well as the quark color labels since they are inessential here. The generators of color $SU(3)$ are denoted by λ^a, and the gluon fields and field strength tensors have been scaled by the bare coupling constant g_0: $\bar{A}_a^\mu = g_0 A_a^\mu$ and $\bar{F}^a_{\mu\nu} = g_0 F^a_{\mu\nu}$.

We first consider the case where the quark mass is set to zero. The grand potential of the system is defined in the usual way, $\Omega = -T \ln Z$, and we have

$$\frac{\partial}{\partial(-1/g_0^2)} \frac{\Omega}{V} = -\frac{g_0^2}{4} \left\langle F_a^{\mu\nu}(0,\mathbf{0}) F^a_{\mu\nu}(0,\mathbf{0}) \right\rangle \equiv -\frac{g_0^2}{4} \left\langle F^2 \right\rangle , \quad (2)$$

where V is the volume of the system. The angle brackets denote a thermal average.

This derivative can be calculated in another way by using an explicit form for Ω/V determined on dimensional grounds. Regulation of the ultraviolet divergences of the theory introduces a mass scale, M, at which the subtractions are performed. The results can be written in terms of the non-perturbative

dimensionful parameter

$$\Lambda = M \exp\left\{ \int\limits_{\alpha_s(M)}^{\infty} \frac{d\alpha_s}{\beta_s(\alpha_s)} \right\} , \tag{3}$$

where $\alpha_s = g_0^2/4\pi$ and β_s is the Gell-Mann-Low function: $M d\alpha_s/dM = \beta_s$. There are two additional dimensionful parameters, namely T and μ. Since the grand potential is an observable quantity with zero anomalous dimension [3], we can write in general

$$\frac{\Omega}{V} = \Lambda^4 f\left(\frac{T}{\Lambda}, \frac{\mu}{\Lambda}\right) , \tag{4}$$

where f is some function. We note that at zero temperature and chemical potential a form proportional to Λ^4 can be formally justified within a well-defined regularization scheme [1]. In Eq. (4) g_0 is involved only through Λ, hence we obtain

$$\frac{\partial}{\partial(-1/g_0^2)} \frac{\Omega}{V} = -\frac{4\pi\alpha_s^2}{\beta_s(\alpha_s)} \left(4 - T\frac{\partial}{\partial T} - \mu\frac{\partial}{\partial \mu}\right) \frac{\Omega}{V} . \tag{5}$$

This leads to the chain of equations

$$\left(4 - T\frac{\partial}{\partial T} - \mu\frac{\partial}{\partial \mu}\right) \frac{\Omega}{V} = \frac{\beta_s(\alpha_s)}{4\alpha_s} \left\langle F^2 \right\rangle = \left\langle \theta_\mu^\mu(0, \mathbf{0}) \right\rangle = \mathcal{E} - 3P . \tag{6}$$

We have used the standard QCD result [4] for the trace of the improved energy-momentum tensor density θ_μ^μ [5]. We have also used the standard thermodynamic relation for the energy density

$$\mathcal{E} = \left(1 - T\frac{\partial}{\partial T} - \mu\frac{\partial}{\partial \mu}\right) \frac{\Omega}{V} , \tag{7}$$

and the pressure $P = -\Omega/V$.

We turn now to the case where the quark mass is non-zero. Since Ω is a physical observable it has to be expressed in terms of renormalization group invariant quantities. However, the quark mass has anomalous dimension and depends on the scale M. The renormalization group equation for $m_0(M)$, the running mass, is $(M/m_0)dm_0/dM = -\gamma_m$ and we use the $\overline{\mathrm{MS}}$ scheme for

which β_s and γ_m are independent of the quark mass [6]. Upon integration the renormalization group invariant mass is given by

$$\hat{m} = m_0(M) \exp \left\{ \int^{\alpha_s(M)} \frac{\gamma_m(\alpha_s)}{\beta_s(\alpha_s)} d\alpha_s \right\} , \tag{8}$$

where the indefinite integral is evaluated at $\alpha_s(M)$. Then Eq. (4) becomes

$$\frac{\Omega}{V} = \Lambda^4 h \left(\frac{T}{\Lambda}, \frac{\mu}{\Lambda}, \frac{\hat{m}}{\Lambda} \right) , \tag{9}$$

where h is some function. Proceeding as before we obtain

$$\begin{aligned} \left(4 - T\frac{\partial}{\partial T} - \mu \frac{\partial}{\partial \mu} \right) \frac{\Omega}{V} &= \frac{\beta_s(\alpha_s)}{4\alpha_s} \left\langle F^2 \right\rangle + [1 + \gamma_m(\alpha_s)]\, m_0 \left\langle \bar{q} q \right\rangle \\ &= \left\langle \theta^\mu_\mu(0, \mathbf{0}) \right\rangle = \mathcal{E} - 3P \; . \end{aligned} \tag{10}$$

Here we have used the trace of the energy-momentum tensor for QCD with quarks [4, 7]. Clearly the physical quantities in Eq. (10) must obey the same relation as in Eq. (6).

We can iterate this procedure by taking n further derivatives of Eq. (10). In doing so it is convenient to note that since the grand potential density and its derivatives are independent of the scale M at which the ultraviolet divergence are regulated, we can choose any scale to prove the result. It is convenient to pick a sufficiently large scale that we can take the lowest order expressions, $\beta_s(\alpha_s) \to -b\alpha_s^2/2\pi$, where $b = (11N - 2N_f)/3$ for the $SU(N)$ gauge group with N_f flavors, and $(1 + \gamma_m) \to 1$. Then it is straightforward to obtain the following relation

$$\begin{aligned} (-1)^n \left(4 - T\frac{\partial}{\partial T} - \mu \frac{\partial}{\partial \mu} \right)^{n+1} \frac{\Omega}{V} &= \left(T\frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - 4 \right)^n \left\langle \theta^\mu_\mu \right\rangle \\ &= \int d\tau_n d^3x_n \cdots \int d\tau_1 d^3x_1 \left\langle \theta^\mu_\mu(\tau_n, \mathbf{x}_n) \cdots \theta^\mu_\mu(\tau_1, \mathbf{x}_1) \theta^\mu_\mu(0, \mathbf{0}) \right\rangle_c \; . \end{aligned} \tag{11}$$

Here the subscript c means that only the connected diagrams are to be included and the limits of the imaginary time integrations are suppressed. It is

possible to perform a similar analysis for a renormalization group invariant operator $\mathcal{O}$

$$\left(T\frac{\partial}{\partial T}+\mu\frac{\partial}{\partial\mu}-d\right)^n\langle\mathcal{O}\rangle$$
$$=\int d\tau_n d^3x_n\cdots\int d\tau_1 d^3x_1\left\langle\theta_\mu^\mu(\tau_n,\mathbf{x}_n)\cdots\theta_\mu^\mu(\tau_1,\mathbf{x}_1)\mathcal{O}(0,\mathbf{0})\right\rangle_c\ , \quad (12)$$

where d is the canonical mass dimension of $\mathcal{O}$. If the operator $\mathcal{O}$ also has anomalous dimension then the corresponding γ-function will have to be included. Equations (11) and (12) differ from those found previously [2] by the addition of the operator $\mu\partial/\partial\mu$. One could also similarly generalize the finite-momentum low-energy theorems [2, 8].

As an example, where we can evaluate the left hand side of Eq. (11), consider the case where μ and/or T is much greater than Λ so that perturbation theory is valid. Considering the first term in the β_s-function, the behavior of the strong coupling constant as a function of chemical potential and temperature can be written

$$\alpha_s(\xi)=\frac{2\pi}{b\ln(\xi/\Lambda)}\ , \quad (13)$$

where $\xi=Ty(\mu/T)$ and we need not specify the function y. The perturbative expression for the pressure in $SU(N)$ gauge theory up to two-loop order is [9]

$$P=\frac{\pi^2}{45}(N^2-1)T^4+N\left(\frac{7\pi^2T^4}{180}+\frac{\mu^2T^2}{6}+\frac{\mu^4}{12\pi^2}\right)$$
$$-\frac{\pi(N^2-1)}{144}\alpha_s(\xi)\left((4N+5)T^4+\frac{18}{\pi^2}\mu^2T^2+\frac{9}{\pi^4}\mu^4\right)\ . \quad (14)$$

Then from Eqs. (13) and (14) we find

$$\left(T\frac{\partial}{\partial T}+\mu\frac{\partial}{\partial\mu}-4\right)^n\left\langle\theta_\mu^\mu\right\rangle=\frac{b}{288}(N^2-1)$$
$$\times\alpha_s^{n+2}(\xi)\left(\frac{-b}{2\pi}\right)^n(n+1)!\left((4N+5)T^4+\frac{18}{\pi^2}\mu^2T^2+\frac{9}{\pi^4}\mu^4\right). \quad (15)$$

This provides a requirement on model or lattice calculations of the right hand side of Eq. (11). Note, however, that if we consider the next order in the

β_s-function then, for $N = 3$ and one flavor in the $\overline{\text{MS}}$-scheme, we obtain a correction factor of order $(1 + 1.4\alpha_s n)$ to Eq. (15). Thus, at this level of approximation for P and α_s, an arbitarily large number of derivatives cannot be taken without at some point losing all accuracy.

In conclusion we have shown that at finite temperature and chemical potential the low-energy theorems of Novikov *et al.* [1] hold provided that the operators $T\partial/\partial T$ and $\mu\partial/\partial\mu$ are appropriately included as in Eqs. (11) and (12).

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